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ON THE STABILITY OF SYSTEMS WITH NON-RETAINING CONSTRAINTS*

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The possibility of extending some of the well-known statements on the stability of the position of equilibrium and of periodic solutions of Hamiltonian systems to systems with ideal non-retaining constraints is pointed out. The problem of equilibrium stability and of periodic jumps of a plane disc moving in a vertical plane above a straight line is considered as an example.

Methods of investigation have been developed which are based on a study of the general properties of motion derived from the behaviour of the system in a finite time interval (see, e.g., /2/). However this presents difficulties in drawing conclusions about such qualitative properties of motion as its stability. A method was proposed in /3/ for obtaining the equations of motion for a system with non-retaining constraints in an arbitrary time interval. It was shown in /4/ that such equations can be expressed in canonical form. These investigations enabled us to extend some of the methods of the theory of stability to systems with non-retaining constraints, and this represents the content of the present paper.

1. Consider a mechanical system in Lagrangian coordinates $\mathbf{q} = (q_1, \ldots, q_n)$ restrained by $v \leqslant n$ non-retaining constraints $f_k(\mathbf{q}) \geqslant 0$ $(k = 1, \ldots, n)$. In the intervals between impacts on constraints, the motion of such systems takes place in accordance with the general principles of mechanics /1/.

We will assume that the coefficients of restitution when there are impacts on the constraints are equal to unity and the Lagrangian function L of system M has the form

$$L = T + U,\ T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\mathbf{q})\, q_i^{\cdot} q_j^{\cdot},\quad U = U(\mathbf{q}),\quad f_\alpha = q_\alpha \geqslant 0 \tag{1.1}$$
$$(\alpha = 1, \ldots, v)$$

where U, a_{ij} are analytic in a certain region D of coordinate space R^n.

Point $A = \mathbf{q}^\circ \in D$ is assumed to be the equilibrium position of system M. We divide the non-retaining constraints into three groups as follows: 1) the constraints $q_1, \ldots, q_{v_1}$ at point A are stressed $- q_\alpha = 0$ and their reactions are non-zero, 2) constraints $q_{v_1+1}, \ldots, q_{v_1+v_2}$ are stressed, but their reactions are zero, and 3) the constraints $q_{v_1+v_2+1}, \ldots, q_n$ are weakened at the point A. An example of such constraints is the equilibrium of a heavy homogeneous sphere on a horizontal plane touching one vertical plane, but at some distance from the other.

By the principle of virtual displacements for systems with non-retaining constraints /1/, the inequality

$$\sum_{i=1}^{n} \frac{\partial U}{\partial q_i}\delta q_i \leqslant 0 \tag{1.2}$$

is satisfied in the equilibrium position.

Assuming that δq_i is non-negative for $i = 1, \ldots, v_1 + v_2$ and can take arbitrary values for $i = v_1 + v_2 + 1, \ldots, n$, while $\partial U/\partial q_i$ is proportional to the reaction of the constraint /1/, we obtain that $\partial U/\partial q_i < 0$ when $i = 1, \ldots, v_1$, and $\partial U/\partial q_1 = 0$ when $i = v_1 + 1, \ldots, n$. The following theorem is a direct extension of the Lagrange-Dirichlet theorem to systems with non-retaining constraints.

Theorem 1. If the function $U(\mathbf{q}^*)$, where $\mathbf{q}^* = (0, \ldots, 0, |q_{v_1+1}|, \ldots, |q_{v_1+v_2}|, q_{v_1+v_2+1}, \ldots, q_n)$, has a strict maximum at the point $A = \mathbf{q}^\circ = (0, \ldots, 0, \overset{\circ}{q}_{v_1+v_2+1}, \ldots, q_n^\circ)$, the position of equilibrium is Lyapunov stable.

Proof. With the assumptions made about the properties of constraints the system has an energy integral $E = T - U$. We shall show that it is positive definite. To do so it is sufficient to ascertain that, when the conditions of the theorem are satisfied, the function $U(\mathbf{q})$ has a strict maximum at the point A in the region $q_i \geqslant 0$ $(i = 1, \ldots, v)$. This statement follows from the fact that the quantity

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$$U(\mathbf{q}) - U(\mathbf{q}^\circ) = U(\mathbf{q}^*) - U(\mathbf{q}^\circ) + \sum_{i=1}^{\nu_1} q_i \frac{\partial U}{\partial q_i}\Big|_{\mathbf{q}^*+\mu(\mathbf{q}-\mathbf{q}^*)}, \quad 0 \leqslant \mu \leqslant 1$$

is negative, when $\mathbf{q} \neq \mathbf{q}^\circ$. Hence the function E satisfies the conditions of the Lyapunov theorem on stability. The theorem is proved.

Let us establish the necessary conditions of stability, when $\nu_1 = 1, \nu_2 = 0$. For this we consider besides the system M, the ancilliary system M_1 with $n-1$ degrees of freedom and the Lagrangian function $L_1 = L(0, q_2, \ldots, q_n, 0, q_2^\cdot, \ldots, q_n^\cdot)$.

Theorem 2. If the equilibrium position $A_1 = (q_2^\circ, \ldots, q_n^\circ)$ of system M_1 is unstable, the equilibrium position $A = (0, q_2^\circ, \ldots, q_n^\circ)$ of system M is also unstable.

Proof. Let us change the variables in system M, using formulas (1.2). We obtain

$$\mathbf{q} = \boldsymbol{\varphi}(\mathbf{Q}), \quad \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n), \quad \mathbf{Q} = (Q_1, \ldots, Q_n)$$

where $\varphi_1 = Q_1$ and the function φ_j represents the solution of the Cauchy problem

$$\sum_{i=2}^{n} a_{ij}(\boldsymbol{\varphi}) \frac{\partial \varphi_i}{\partial Q_1} = -a_{1j}(\boldsymbol{\varphi}), \quad \varphi_j|_{Q_1=0} = Q_j \quad (j = 2, \ldots, n) \tag{1.3}$$

In new variables the Lagrangian function (1.1) does not contain the products $Q_1^\cdot Q_k^\cdot$ $(k = 2, \ldots, n)$ /4/. The equations of motion take the form

$$a_{11} q_1^{\cdot\cdot} + a_{11}^\cdot q_1^\cdot - \frac{\partial L}{\partial q_1} = F_1, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial q_k^\cdot}\right) - \frac{\partial L}{\partial q_k} = 0 \quad (k = 2, \ldots, n) \tag{1.4}$$

where $F_1 = 0$ when $q_1 \neq 0$, and when $q_1 = q_1^\cdot = 0$ the function F_1 is defined so that the generalized acceleration $q_1^{\cdot\cdot}$ is transformed into a possible one; it can be shown that $F_1 = \max\{0, -\partial L/\partial q_1\}$. The second group of equations (1.4) does not contain $q_1^{\cdot\cdot}$.

Since $\partial U/\partial q_1|_A < 0$, a neighbourhood V of a point of the phase space $(A, 0)$ can be found at which $\partial L/\partial q_1 < 0$. In the V region the first of equations (1.4) can be satisfied by setting $q_1 \equiv 0$. Then the second group of these equations represent the equations of motion of system M_1, and the trajectories of that system in region V are simultaneously the trajectories of system M_1 for which $q_1 = 0$, from which the theorem follows.

Corollary. Let us represent the function $U(\mathbf{q})$ in the neighbourhood of the point $A = \mathbf{q}^\circ$ in the form of the series

$$U = Rq_1 + U_m + U_{m+1} + \ldots, \quad m \geqslant 2 \tag{1.5}$$

where U_m is a homogeneous polynomial of power m of $(\mathbf{q} - \mathbf{q}^\circ)$ and $R = \partial U/\partial q_1|_A < 0$. If for $q_1 = 0$ the function is negatively defined (may take positive values), then the equilibrium position of A is Lyapunov stable (unstable).

The statement about instability is based here n the results obtined in /6/.

2. Let us now investigate the stability of periodic motions of system (1.1) consisting of sections on which the constraints $q_i \geqslant 0$ $(i = 1, \ldots, \nu)$ are weakened and impacts on the constraint are $q_1 \geqslant 0$. Motions of this kind were considered in the theory of vibro-impact systems /2/. The basic technical difficulty in investigating such systems is that due to impact interaction the generalized velocities are discontinuous functions of time, and in the conventional definition the perturbed motion is determined by equations whose right sides are discontinuous functions of the perturbations /5/ (see /3/). The possibility of regularizing the equations of perturbed motion, when introducing the canonical formalism into (1.1), is shown in /4/. The general procedure for obtaining the perturbed motion in the form of an analytic function of perturbations for solving this type of Hamiltonian is shown below.

Using the replacement (1.2), we select the generalized coordinates so as to obtain in (1.1) $a_{1k} \equiv 0$ $(k = 2, \ldots, n)$, and we determine the motion of system M using the ancilliary system M^* which is free of the constraint $q_1 \geqslant 0$ and has the Lagrangian function $L^*(\mathbf{q}, \mathbf{q}^*) = L(|q_1|, q_2, \ldots, q_n, \mathbf{q}^\cdot)$. For the trajectories $\mathbf{q}(t)$ and $\mathbf{q}^*(t)$ of systems M and M^* the following relations are satisfied /4/:

$$q_1(t) = |q_1^*(t)|, \quad q_i(t) = q_i^*(t) \quad (i = 2, \ldots, n)$$

which enables us to establish the equivalence of systems M and M^* from the point of view of the stability of their partial solutions.

Setting

$$p_k = \frac{\partial L^*}{\partial q_k^\cdot}, \quad H = \sum_{k=1}^{n} q_k^\cdot p_k - L^*$$

we write the equations of motion of system M^* in the canonical form

$$\frac{d\mathbf{q}}{dt}=\frac{\partial H}{\partial \mathbf{p}},\quad \frac{d\mathbf{p}}{dt}=-\frac{\partial H}{\partial \mathbf{q}},\quad \left.\frac{\partial H}{\partial q_1}\right|_{q_1=0}=\min\left\{0,\left.\frac{\partial H}{\partial q_1}\right|_{q_1=+0}\right\} \tag{2.1}$$

Since the Hamiltonian function H does not depend explicitly on time, it represents the integral of motion in the sense of the total mechanical energy E. Let us assume that for E taken from some interval containing the point E_0, system (2.1) has a particular solutions of the form

$$q_1 = q_1^\circ(t, E),\quad p_1 = p_1^\circ(t, E),\quad q_k = q_k^\circ,\quad p_k = p_k^\circ \quad (k = 2, \ldots, n) \tag{2.2}$$

where the functions q_1°, p_1° are solutions of the system

$$q_1^{\cdot}=\frac{\partial H^\circ}{\partial p_1},\quad p_1^{\cdot}=-\frac{\partial H^\circ}{\partial q_1},\quad H^\circ=H(q_1, q_2^\circ, \ldots, q_n^\circ, p_1, p_2^\circ, \ldots, p_n^\circ) \tag{2.3}$$

which is t periodic of period $\tau(E)$. We denote the zeros of the function $q_1^\circ(t, E)$ in a period by $t_1(E)\leqslant t_2(E)\leqslant\ldots\leqslant t_k(E)=t_1(E)+\tau(E)$. We assume that the following relations are satisfied:

$$\frac{t_i(E)-t_1(E)}{t_i(E_0)-t_1(E_0)}=\frac{T(E)}{T(E_0)}\quad (i=2,\ldots,k) \tag{2.4}$$

We change in system (2.3) to "action-angle" variables, using the formulas

$$q_1=Q(w, I)=q_1^\circ\left(\frac{w}{2\pi}\tau(s), s\right),\quad p_1=P(w, I)=p_1^\circ\left(\frac{w}{2\pi}\tau(s), s\right) \tag{2.5}$$

where q_1°, p_1° were defined in (2.2) and $s = s(I)$ is the solution of the differential equation

$$\frac{ds}{dI}=\frac{2\pi}{\tau(s)},\quad s(I_0)=E_0$$

The change (2.5) is canonical and periodic in w of period 2π.
When $E = E_0$, the solution of (2.2) takes the form

$$I=I_0,\quad w=w_0+\frac{2\pi}{\tau(E_0)}t,\quad q_k=q_k^\circ,\quad p_k=p_k^\circ\quad (k=2,\ldots,n) \tag{2.6}$$

Let us consider the problem of the stability of solution (2.6) with respect to perturbations of the variables q_k, p_k $(k = 2, \ldots, n)$ and the action variable I.

Since by (2.4) the disposition of the zeros of the function $Q(w, I)$, as defined in (2.5), is independent of I, the relations

$$Q(w, I) = Q(w, I_0)\cdot F(w, I),\quad F(w, I)>0 \tag{2.7}$$

holds and the quantity $|q_0|$ in the expression for the Hamiltonian H is by virtue of the change (2.5) an analytic function of I when $I = I_0$. The presence of impact interaction in the motion leads only to lack of smoothness with respect to the variable w. This does not, however, prevent its application to the solution of the above problem of the stability of algorithms of the analytic theory of perturbations (see /7, 8/).

3. An example, we shall consider the motion of a heavy plane disc in the upper half of some vertical plane. Let OXY and $O'X'Y'$ be systems of coordinates in the plane of motion and attached to the disc, respectively; the OX axis coincides with the straight line bounding the motion of the disc, OY is a vertical line, and O' is at the disc centre of mass. The curvilinear boundary of the disc may be specified by the analytic function $f(\alpha)$ whose value is equal to the distance from the point O' to the tangent to that curve which makes an angle α with the $O'X'$ axis.

As the Lagrangian coordinates we take $q_1 = y - f(\alpha)$, $q_2 = \alpha$, where x, y are the coordinates of point O' in the system OXY and α is the angle between OX and $O'X'$. The Lagrangian function has the form

$$L=T+U,\ T=\frac{m}{2}\{[q_1^{\cdot}+f'(q_2)q_2^{\cdot}]^2+q_3^{\cdot 2}\}+\frac{F}{2}q_2^{\cdot 2},\quad U=-mg[q_1+f(q_2)] \tag{3.1}$$

where m and J are, respectively, the mass of disc and its moment of inertia about the point O', and g is the free fall acceleration.

Note that the variable q_3 is cyclic, and in the Lagrange equations this variable is separated from q_1, q_2. Hence the coordinate $q_3 = x$ is a linear function of time, which is a corollary of the fact that the horizontal components of the force of gravity and the impact forces are zero. Below we shall investigate the behaviour of the variables q_1, q_2.

In the coordinate space q_1, q_2 the points $q_1 = 0, q_2 = a$ correspond to the equilibrium positions, when $f'(a) = 0$. If the disc is not a circle with the centre of gravity at its geometric centre, then because of the assumption made about the analyticity of $f(\alpha)$, these positions are isolated.

On the assumptions made in Sect.1, it is necessary and sufficient for the stability of these equilibrium positions that the point a be the minimum of the function $f(\alpha)$.

The equilibrium positions are points that generate sets of periodic motions which are jumps with a constant value of the coordinate q_2 equal to a. To investigate the stability of such solutions we apply the procedure described in Sect.2.

Since in the Lagrangian function (3.1) the coefficient of $q_1^{\cdot}, q_2^{\cdot}$ is non-zero, to obtain the canonical form of the equations of motion it is necessary to use the reducing change of variables (1.2) which has here the form $q_1 = Q_1$, $q_2 = \varphi(Q_1, Q_2)$, where the function φ is the solution of the Cauchy problem

$$\frac{\partial\varphi}{\partial Q_1} = -G(\varphi), \quad \varphi|_{Q_1=0} = Q_2, \quad G = \frac{mf'}{J + mf'^2} \tag{3.2}$$

The Lagrangian (3.1) in the new variables has the form

$$L = \frac{mD(\varphi)}{2} Q_1^{\cdot 2} + \frac{J\bar{\varphi}^2}{2D(\varphi)} Q_2^{\cdot 2} - [Q_1 + f(\varphi)]mg, \quad Q_1 \geqslant 0$$

$$D = \frac{J}{J + mf'^2}, \quad \bar{\varphi} = \frac{\partial\varphi}{\partial Q_2}$$

The equations of motion of the ancilliary system M^* have the canonical form (2.1) with a Hamiltonian function of the form

$$H = \frac{1}{2mD(\psi)} P_1^2 + \frac{D(\psi)}{2J\bar{\psi}^2} P_2^2 + [|Q_1| + f(\psi)]mg, \quad \psi = \varphi(|Q_1|, Q_2), \quad \bar{\psi} = \frac{\partial\psi}{\partial Q_2} \tag{3.3}$$

The periodic motions of the disc mentioned above correspond to particular solutions of system (2.1), (3.3) of the form

$$\bar{Q}_1 = \frac{t}{2}\left[2\left(\frac{2E}{m}\right)^{1/2} - g|t|\right], \quad \bar{Q}_2 = a, \quad \bar{P}_1 = (2Em)^{1/2} - |t|mg, \tag{3.4}$$

$$\bar{P}_2 = 0$$

$$-\frac{\tau}{2} \leqslant t \leqslant \frac{\tau}{2}, \quad \tau = \frac{4}{g}\left(\frac{2E}{m}\right)^{1/2}$$

where τ is the period of the motion considered, (i.e. the time interval between the k-th and the $(k+2)$-th impacts, $k = 1, 2, \ldots$), and E is the energy constant. Formulas (2.5) of passing to "action-angle" variables takes the form

$$Q_1 = \frac{2}{gm^2}\left(\frac{3gm^2}{2\pi^2} I\right)^{2/3} w(\pi - |w|), \quad P_1 = 2\left(\frac{3gm^2}{2\pi^2} I\right)^{1/3}\left(\frac{\pi}{2} - |w|\right) \tag{3.5}$$

$$-\pi \leqslant w \leqslant \pi$$

The change is periodic in w of period 2π.

Solutions (3.4) become

$$I = I_0, \quad w = w_0 + \frac{mg}{2}\left(\frac{3gm^2}{2\pi^2} I_0\right)^{-1/3} t, \quad Q_2 = a, \quad P_2 = 0$$

We define the perturbed motion in variables ξ, η, r which are defined as follows:

$$\xi = Q_2 - a, \quad \eta = P_2, \quad r = I - I_0 \tag{3.6}$$

and represent the Hamiltonian (3.3) and the function ψ in the form of power series

$$H = H_2 + \ldots + H_m + \ldots \tag{3.7}$$

$$H_2 = h_{002}r + H_2^{\circ}, \quad H_2^{\circ} = h_{200}\xi^2 + h_{020}\eta^2$$

$$H_m = \sum_{\nu+\mu+\rho=m} h_{\nu,\mu,\rho}(w)\, \xi^{\nu}\eta^{\mu}r^{\rho/2}, \quad h(w + 2\pi) = h(w)$$

$$h_{200} = \frac{mf''(a)\psi_1^2}{2}\left[g + \frac{f''(a)}{J}\bar{P}_1^2\right], \quad h_{020} = \frac{1}{2J\psi_1^2}$$

$$h_{002} = \frac{\pi g}{2}\left(\frac{m}{2E_0}\right)^{1/2}, \quad E_0 = \frac{\pi^2}{2m}\left[\frac{3m^2g}{2\pi^2} I_0\right]^{2/3}$$

$$\psi = a + \psi_1(Q_1^{\circ})\,\xi + \psi_2(Q_1^{\circ})\,\xi^2 + \ldots \tag{3.8}$$

The function ψ_k in formula (3.8) may be determined by expanding $G(\psi)$ in series in powers of $(\psi - a)$, and substituting (3.8) into (3.2). We successively obtain

$$\psi_1 = \exp\left(-\frac{f''(a)m}{J}|Q_1^{\circ}|\right), \quad \psi_2 = \frac{f'''(a)}{2f''(a)}(\psi_1^2 - \psi_1) \tag{3.9}$$

etc, where Q_1° is defined by (3.5) when $I = I_0$, in which case $Q_1^{\circ} = \bar{Q}_1$.

According to the general method of reduction to normal form /9/, we consider first the linear system with the Hamiltonian H_2° and the idependent variable w. To construct the fundamental matrix of solutions $X(t)$ we may use the following fairly simple reasoning.

The quantity $q_2 = \alpha$ in the intervals between impacts varies according to the well-known law (linear, since the moment of the force of gravity about the point O' is zero). The variables q_2 and ξ are connnected by the relation $q_2 = a + \psi_1(Q_1^\circ)\xi + O(\xi^2, r)$. From this we can determine the solution for ξ, η in the interval between impacts. Since the variable $\eta = P_2$ remains continuous on impact /10/, that solution is uninterruptedly continued during the impact, and the need to apply the method of adjustment is eliminated.

Calculations of $X(w)$ in the interval $[0, \pi]$ yield the expression

$$X(w) = |x_{ij}| \ (i, j = 1, 2), \ x_{11} = (1 + \varkappa_0\overline{w})/\psi_1 \tag{3.10}$$

$$x_{12} = \overline{w}/(J\psi_1), \ x_{21} = J\psi_1[\varkappa_0 - \varkappa(1 + \varkappa_0\overline{w})], \ x_{22} = \psi_1(1 - \varkappa\overline{w})$$

$$\overline{w} = \frac{2}{\pi g}\left(\frac{2E_0}{m}\right)^{1/2} w, \quad \varkappa = \varkappa(w) = -\frac{f''(a)}{J} P_1^\circ(w), \quad \varkappa_0 = \varkappa(0)$$

where P_1° is determined using (3.5) when $I = I_0$ and ψ_1 is given by (3.9).

Noting that the coefficients of the function H_2° are π-periodic functions of w, we can write the characteristic equation in the form

$$\det\|X(\pi) - \rho E_2\| = \rho^2 - (2 - 8k)\rho + 1 = 0, \qquad k = \frac{m}{J} hf''(a)$$

and h is the height of jump in the periodic motion. Solving this equation we obtain the necessary condition of stability

$$0 < k < 1/2 \tag{3.11}$$

and determine the multipliers $\rho_{1,2} = \exp(\pm\pi i\lambda)$.

Note that the quantity $f''(a)$ represents the difference between the radius of curvature of the disc at the point with $q_2 = a$ and the distance of that point from the point O'. When the disc is an inhomogeneous circle of radius R, in which the distance between the geometric centre and the centre of gravity is ε, condition (3.11) means that in periodic motion the centre of mass is below the geometric centre, and $mJ^{-1}h\varepsilon < 1/2$. In the case of a homogeneous disc in the form of an ellipse with semiaxes $a_1 > a_2$, condition (3.11) means that the impacts occur along the a_2 axis and $h < a_2(a_1^2 + a_2^2)/[8(a_1^2 - a_2^2)]$.

We shall investigate the stability in a non-linear formulation by analysing the forms H_3, H_4 in the expansion (3.7).

When $k = 3/8$ the characteristic index λ is related to the frequency of periodic motion $\omega = 2$ by the third-order resonance relation $3\lambda = \omega$. The calculations show that, if $f'''(a) \neq 0$, the periodic motion is unstable.

For the remaining values of k from the interval $(0, 1/2)$ the question of stability depends on the parameters

$$\varkappa_1 = \left(\frac{J}{m}\right)^{1/2} \frac{f'''(a)}{[f''(a)]^2}, \quad \varkappa_2 = \frac{J}{m}\frac{f^{IV}(a)}{[f''(a)]^3}$$

when $k = 1/4$ (there is no fourth-order resonance $4\lambda = \omega$) and the normal form is non-degenerate, in general we have stability of the motion considered.

The calculation carried out for an inhomogeneous disc and for a homogeneous ellipse ($\varkappa_1 = 0$) show that in these cases solutions (3.4) are stable for all values of k from the interval $(0, 1/2)$.

4. Let us now turn to the more general case when the Lagrangian function has the form

$$L' = T' + U', \ T' = T_2 + T_1 + T_0, \ U' = U'(\mathbf{q}), \ q_i \geqslant 0 \tag{4.1}$$

$$(i = 1, \ldots, \nu)$$

where the coefficients of the form $T_i \ (i = 0, 1, 2)$ are independent of time.

Since system (4.1) admits of the energy integral $E = T_2 - T_0 - U'$, for the stability of its equilibrium position it is sufficient that the conditions of Theorem 1 for the function $U = T_0 + U'$ be satisfied. Theorem 2 also remains valid. The corollary of that theorem in the form given above is generally incorrect, since the reduced system M_1 in the conditions indicated may prove to be stable owing to the gyroscopic forces generated by T_1.

To derive the equations of motion of system (4.1) when $\nu = 1$ we shall take advantage of the property of the dynamic equivalence of that system to system M of the form (1.1) with $n = k + 1$ and can be obtained from it by ignoring the cyclic coordinate q_n /11/. We apply to system M a reducing change, using formulas (1.2) in which φ_i are independent of $Q_n \ (i = 2, \ldots, n-1)$. Since the coefficients a_{ij} are independent of q_n, such a solution of the Cauchy problem (1.3) exists, and $q_n = \varphi_n = Q_n + F(Q_1, \ldots, Q_{n-1})$. Substituting the functions $\varphi, \varphi^\cdot$ for $\mathbf{q}, \mathbf{q}^\cdot$ into (1.1), we obtain an expression for the Lagrangian function in terms of new variables in which the products $Q_1^\cdot Q_i^\cdot \ (i = 2, \ldots, n)$ have vanished and the variable Q_n is cyclic. Passing as in Sect.2 to the ancilliary system M^*, using in the Lagrangian the substitution $Q_1 \to |Q_1|$

and, then, eliminating the variable Q_n, we can finally obtain the equations of motion of system (4.1) in the from (2.1).

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THE APPLICATION OF ASYMPTOTIC METHODS TO CERTAIN STOCHASTIC PROBLEMS OF THE DYNAMICS OF VIBROPERCUSSIVE SYSTEMS*

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Motion of certain vibropercussive systems acted upon by a random, non-white noise perturbation is studied, using the limit theorems of the convergence of the solutions of stochastic differential equations to a diffusion process. The results, first obtained in /1, 2/ for smooth systems, are generalized to include systems with discontinuous and impulsive right-hand sides /3–5/ by approximating the discontinuous functions by a converging sequence of smooth functions. An analogous approach is described for vibropercussive systems, and regions of stability of the perturbed motion are constructed.

Analytic expressions describing the probability density and dispersions of velocity and coordinates are well known /6, 7/ in the case of linear systems excited by white noise, under elastic impact. The method of non-smooth transformations /8/ is used for more complex systems to construct the FPK equations characterizing the distribution of the energy of the oscillations /9, 10/. Basic results are also obtained for systems excited by white noise.

1. Consider a quasiconservative, vibropercussive system. The equation of motion and condition of impact against a one-sided stop have the form

$$x^{\cdot\cdot} + \Omega^2 x = \varepsilon g(t, x, x^{\cdot}, \varepsilon) \tag{1.1}$$

$$x = \Delta,\ x_+^{\cdot} = -Rx_-^{\cdot},\ R = 1 - \varepsilon^2 r,\ r = \text{const} = O(1) \tag{1.2}$$

Here Δ is the size of the gap $(\Delta > 0)$ or displacement $(\Delta < 0)$, $x_-^{\cdot}$ and $x_+^{\cdot}$ denote the velocities before and after the impact and ε is a small parameter. The piecewise-continuous function g characterizes the additional non-conservative terms and represents, for fixed x

*Prikl.Matem.Mekhan.,48,5,733-737,1984